mhd-hermes Documentation

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ONE

INSTALLATION INSTRUCTIONS

Install hermes2d, so that you can `import hermes2d` from Python:

```
In [1]: import hermes2d

In [2]:
```

Once this works, then just run:

```
cmake .
make
```

and that's it (cmake will ask the `hermes2d` module where all the `*.h` and `*.pxd` files are).

CHAPTER

TWO

MHD EQUATIONS

2.1 Introduction

The magnetohydrodynamics (MHD) equations are:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{2.1}$$

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B} + \rho\mathbf{g} \tag{2.2}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2\mathbf{B} \tag{2.3}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{2.4}$$

assuming η is constant. See the next section for a derivation. We can now apply the following identities (we use the fact that $\nabla\cdot\mathbf{B}=0$):

$$[(\nabla\times\mathbf{B})\times\mathbf{B}]_i = \varepsilon_{ijk}(\nabla\times\mathbf{B})_j B_k = \varepsilon_{ijk}\varepsilon_{jlm}(\partial_l B_m)B_k = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(\partial_l B_m)B_k =$$
$$= (\partial_k B_i)B_k - (\partial_i B_k)B_k = \left[(\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{1}{2}\nabla|\mathbf{B}|^2\right]_i$$
$$(\nabla\times\mathbf{B})\times\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{1}{2}\nabla|\mathbf{B}|^2 = (\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{B}(\nabla\cdot\mathbf{B}) - \frac{1}{2}\nabla|\mathbf{B}|^2 = \nabla\cdot(\mathbf{B}\mathbf{B}^T) - \frac{1}{2}\nabla|\mathbf{B}|^2$$
$$\nabla\times(\mathbf{v}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{v} - \mathbf{B}(\nabla\cdot\mathbf{v}) + \mathbf{v}(\nabla\cdot\mathbf{B}) - (\mathbf{v}\cdot\nabla)\mathbf{B} = \nabla\cdot(\mathbf{B}\mathbf{v}^T - \mathbf{v}\mathbf{B}^T)$$
$$\nabla\cdot(\rho\mathbf{v}\mathbf{v}^T) = (\nabla\cdot(\rho\mathbf{v}))\,\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\mathbf{v}\frac{\partial\rho}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v}$$

So the MHD equations can alternatively be written as:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{2.5}$$

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v}^T) = -\nabla p + \frac{1}{\mu}\left(\nabla\cdot(\mathbf{B}\mathbf{B}^T) - \frac{1}{2}\nabla|\mathbf{B}|^2\right) + \rho\mathbf{g} \tag{2.6}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\cdot(\mathbf{B}\mathbf{v}^T - \mathbf{v}\mathbf{B}^T) + \eta\nabla^2\mathbf{B} \tag{2.7}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{2.8}$$

One can also introduce a new variable $p^* = p + \frac{1}{2}\nabla|\mathbf{B}|^2$, that simplifies (2.6) a bit.

2.2 Derivation

The above equations can easily be derived. We have the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Navier-Stokes equations (momentum equation) with the Lorentz force on the right-hand side:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}$$

where the current density $\mathbf{j}$ is given by the Maxwell equation (we neglect the displacement current $\frac{\partial \mathbf{E}}{\partial t}$):

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B}$$

and the Lorentz force:

$$\frac{1}{\sigma} \mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

from which we eliminate $\mathbf{E}$:

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma} \mathbf{j} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma \mu} \nabla \times \mathbf{B}$$

and put it into the Maxwell equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

so we get:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \left(\frac{1}{\sigma \mu} \nabla \times \mathbf{B} \right)$$

assuming the magnetic diffusivity $\eta = \frac{1}{\sigma \mu}$ is constant, we get:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \eta \nabla \times (\nabla \times \mathbf{B}) = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \left(\nabla^2 \mathbf{B} - \nabla (\nabla \cdot \mathbf{B}) \right) = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

where we used the Maxwell equation:

$$\nabla \cdot \mathbf{B} = 0$$

2.3 Finite Element Formulation

We solve the following ideal MHD equations (we use $p^* = p + \frac{1}{2} \nabla |\mathbf{B}|^2$, but we drop the star):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla p = 0 \tag{2.9}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} = 0 \tag{2.10}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.11}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.12}$$

If the equation (2.12) is satisfied initially, then it is satisfied all the time, as can be easily proved by applying a divergence to the Maxwell equation $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$ and we get $\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0$, so $\nabla \cdot \mathbf{B}$ is constant, independent of time. As a consequence, we are essentially only solving equations (2.9), (2.10) and (2.11), which consist of 5 equations for 5 unknowns (components of $\mathbf{u}$, p and $\mathbf{B}$).

We discretize in time by introducing a small time step τ and we also linearize the convective terms:

$$\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^n - (\mathbf{B}^{n-1} \cdot \nabla)\mathbf{B}^n + \nabla p = 0 \tag{2.13}$$

$$\frac{\mathbf{B}^n - \mathbf{B}^{n-1}}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla)\mathbf{B}^n - (\mathbf{B}^{n-1} \cdot \nabla)\mathbf{u}^n = 0 \tag{2.14}$$

$$\nabla \cdot \mathbf{u}^n = 0 \tag{2.15}$$

Testing (2.13) by the test functions (v_1, v_2), (2.14) by the functions (C_1, C_2) and (2.15) by the test function q, we obtain the following weak formulation:

$$\begin{aligned}
\int_\Omega \frac{u_1 v_1}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) u_1 v_1 - (\mathbf{B}^{n-1} \cdot \nabla) B_1 v_1 - p \frac{\partial v_1}{\partial x} \,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{u_1^{n-1} v_1}{\tau} \,\mathrm{d}\mathbf{x} \\
\int_\Omega \frac{u_2 v_2}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) u_2 v_2 - (\mathbf{B}^{n-1} \cdot \nabla) B_2 v_2 - p \frac{\partial v_2}{\partial y} \,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{u_2^{n-1} v_2}{\tau} \,\mathrm{d}\mathbf{x}
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\int_\Omega \frac{B_1 C_1}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) B_1 C_1 - (\mathbf{B}^{n-1} \cdot \nabla) u_1 C_1 \,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{B_1^{n-1} C_1}{\tau} \,\mathrm{d}\mathbf{x} \\
\int_\Omega \frac{B_2 C_2}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) B_2 C_2 - (\mathbf{B}^{n-1} \cdot \nabla) u_2 C_2 \,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{B_2^{n-1} C_2}{\tau} \,\mathrm{d}\mathbf{x}
\end{aligned} \tag{2.17}$$

$$\int_\Omega \frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \,\mathrm{d}\mathbf{x} = 0 \tag{2.18}$$

To better understand the structure of these equations, we write it using bilinear and linear forms, as well as take into account the symmetries of the forms. Then we get a particularly simple structure:

$$\begin{array}{llllllll}
+A(u_1, v_1) & & -X(p, v_1) & -B(B_1, v_1) & & = & l_1(v_1) \\
& +A(u_2, v_2) & -Y(p, v_2) & & -B(B_2, v_2) & = & l_2(v_2) \\
+X(q, u_1) & +Y(q, u_2) & & & & = & 0 \\
-B(u_1, C_1) & & & +A(B_1, C_1) & & = & l_4(C_1) \\
& -B(u_2, C_2) & & & +A(B_2, C_2) & = & l_5(C_2)
\end{array}$$

where:

$$
\begin{aligned}
A(u,v) &= \int_\Omega \frac{uv}{\tau} + (\mathbf{u}^{n-1}\cdot\nabla)uv \,\mathrm{d}\mathbf{x} \\
B(u,v) &= \int_\Omega (\mathbf{B}^{n-1}\cdot\nabla)uv \,\mathrm{d}\mathbf{x} \\
X(u,v) &= \int_\Omega u\frac{\partial v}{\partial x} \,\mathrm{d}\mathbf{x} \\
Y(u,v) &= \int_\Omega u\frac{\partial v}{\partial y} \,\mathrm{d}\mathbf{x} \\
l_1(v) &= \int_\Omega \frac{u_1^{n-1}v}{\tau} \,\mathrm{d}\mathbf{x} \\
l_2(v) &= \int_\Omega \frac{u_2^{n-1}v}{\tau} \,\mathrm{d}\mathbf{x} \\
l_4(v) &= \int_\Omega \frac{B_1^{n-1}v}{\tau} \,\mathrm{d}\mathbf{x} \\
l_5(v) &= \int_\Omega \frac{B_2^{n-1}v}{\tau} \,\mathrm{d}\mathbf{x}
\end{aligned}
$$

E.g. there are only 4 distinct bilinear forms. Schematically we can visualize the structure by:

A		-X	-B	
	A	-Y		-B
X	Y			
-B			A	
	-B			A

In order to solve it with Hermes, we first need to write it in the block form:

$$
\begin{array}{lllllllllll}
a_{11}(u_1,v_1) & + & a_{12}(u_2,v_1) & + & a_{13}(p,v_1) & + & a_{14}(B_1,v_1) & + & a_{15}(B_2,v_1) & = & l_1(v_1) \\
a_{21}(u_1,v_2) & + & a_{22}(u_2,v_2) & + & a_{23}(p,v_2) & + & a_{24}(B_1,v_2) & + & a_{25}(B_2,v_2) & = & l_2(v_2) \\
a_{31}(u_1,q) & + & a_{32}(u_2,q) & + & a_{33}(p,q) & + & a_{34}(B_1,q) & + & a_{35}(B_2,q) & = & l_3(q) \\
a_{41}(u_1,C_1) & + & a_{42}(u_2,C_1) & + & a_{43}(p,C_1) & + & a_{44}(B_1,C_1) & + & a_{45}(B_2,C_1) & = & l_4(C_1) \\
a_{51}(u_1,C_2) & + & a_{52}(u_2,C_2) & + & a_{53}(p,C_2) & + & a_{54}(B_1,C_2) & + & a_{55}(B_2,C_2) & = & l_5(C_2)
\end{array}
$$

comparing to the above, we get the following nonzero forms:

$$
\begin{array}{lllllllllll}
a_{11}(u_1,v_1) & + & 0 & + & a_{13}(p,v_1) & + & a_{14}(B_1,v_1) & + & 0 & = & l_1(v_1) \\
0 & + & a_{22}(u_2,v_2) & + & a_{23}(p,v_2) & + & 0 & + & a_{25}(B_2,v_2) & = & l_2(v_2) \\
a_{31}(u_1,q) & + & a_{32}(u_2,q) & + & 0 & + & 0 & + & 0 & = & 0 \\
a_{41}(u_1,C_1) & + & 0 & + & 0 & + & a_{44}(B_1,C_1) & + & 0 & = & l_4(C_1) \\
0 & + & a_{52}(u_2,C_2) & + & 0 & + & 0 & + & a_{55}(B_2,C_2) & = & l_5(C_2)
\end{array}
$$

where:

$$
\begin{aligned}
a_{11}(u_1, v_1) &= A(u_1, v_1) \\
a_{22}(u_2, v_2) &= A(u_2, v_2) \\
a_{44}(B_1, C_1) &= A(B_1, C_1) \\
a_{55}(B_2, C_1) &= A(B_2, C_2) \\
a_{13}(p, v_1) &= -X(p, v_1) \\
a_{31}(u_1, q) &= X(q, u_1) \\
a_{23}(p, v_2) &= -Y(p, v_2) \\
a_{32}(u_2, q) &= Y(q, u_2) \\
a_{14}(B_1, v_1) &= -B(B_1, v_1) \\
a_{41}(u_1, C_1) &= -B(u_1, C_1) \\
a_{25}(B_2, v_2) &= -B(B_2, v_2) \\
a_{52}(u_2, C_2) &= -B(u_2, C_2)
\end{aligned}
$$

and $l1$, ..., $l5$ are the same as above.

CHAPTER

THREE

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